

Optimal transport and shape interpolation

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Abstract. We propose a method to interpolate between two shapes based on optimal transport. It remains stable when the topologies differ, and its computational cost grows linearly with the number of points.

1. Introduction

Comparing two shapes often amounts to measuring the effort needed to move one onto the other. Optimal transport gives this idea a precise framework: we look for the map that moves the mass at the lowest cost.

Classical interpolation methods, based on a point-to-point correspondence, fail as soon as the shapes do not share the same topology. Optimal transport, on the other hand, assumes no correspondence: it computes it.

2. Formulation

Let μ and ν be two probability measures on $\mathbb{R}^d$. The Monge problem looks for a map T that pushes μ onto ν :

$$\min_{T_{\#}\mu=\nu} \int_{\mathbb{R}^d} \|x - T(x)\|^2 d\mu(x). \quad (1)$$

Its relaxation, due to Kantorovich, is over the couplings π with marginals μ and ν :

$$W_2^2(\mu, \nu) = \min_{\pi \in \Pi(\mu, \nu)} \int \|x - y\|^2 d\pi(x, y). \quad (2)$$

Theorem (Brenier). *If μ is absolutely continuous, problem Equation 2 has a unique solution, supported on the graph of $T = \nabla \varphi$, where φ is a convex function.*

McCann's interpolation then continuously connects the two shapes:

$$\mu_t = ((1 - t) \text{Id} + tT)_{\#}\mu, \quad t \in [0, 1]. \quad (3)$$

3. Computation

We regularize the problem with entropy, which leads to the Sinkhorn algorithm. With $K = e^{-C/\varepsilon}$, the iterations read

$$a^{(k+1)} = \frac{\mu}{Kb^{(k)}}, \quad b^{(k+1)} = \frac{\nu}{K^{\top}a^{(k+1)}}, \quad (4)$$

and the coupling is given by $\pi = \text{diag}(a)K \text{diag}(b)$. Each iteration only requires matrix-vector products.

4. Results

On all our examples, the method preserves fine details while avoiding the aliasing artifacts observed with classical approaches.

Method	Points	Time (s)	Error
Direct correspondence	10^4	0.8	0.112

Sinkhorn (ours)	10^4	1.3	0.031
Sinkhorn (ours)	10^5	12.9	0.029

5. Conclusion

Optimal transport provides a natural interpolation between shapes of different topologies. We will next study the case of animated shapes.

References

C. Villani, *Optimal Transport: Old and New*, Springer, 2009.

M. Cuturi, Sinkhorn Distances, *NeurIPS*, 2013.