

Optimal transport and shape interpolation

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Abstract

We propose a method to interpolate between two shapes based on optimal transport. It remains stable when the topologies differ, and its computational cost grows linearly with the number of points.

1 Introduction

Comparing two shapes often amounts to measuring the effort needed to move one onto the other. Optimal transport gives this idea a precise framework: we look for the map that moves the mass at the lowest cost.

Classical interpolation methods, based on a point-to-point correspondence, fail as soon as the shapes do not share the same topology. Optimal transport, on the other hand, assumes no correspondence: it computes it.

2 Formulation

Let μ and ν be two probability measures on $\mathbb{R}^d$. The Monge problem looks for a map T that pushes μ onto ν :

$$\min_{T_{\#}\mu=\nu} \int_{\mathbb{R}^d} \|x - T(x)\|^2 d\mu(x). \quad (1)$$

Its relaxation, due to Kantorovich, is over the couplings π with marginals μ and ν :

$$W_2^2(\mu, \nu) = \min_{\pi \in \Pi(\mu, \nu)} \int \|x - y\|^2 d\pi(x, y). \quad (2)$$

Theorem 1 (Brenier). *If μ is absolutely continuous, problem (2) has a unique solution, supported on the graph of $T = \nabla\varphi$, where φ is a convex function.*

McCann's interpolation then continuously connects the two shapes:

$$\mu_t = ((1 - t) \text{Id} + tT)_{\#}\mu, \quad t \in [0, 1]. \quad (3)$$

3 Computation

We regularize the problem with entropy, which leads to the Sinkhorn algorithm. With $K = e^{-C/\varepsilon}$, the iterations read

$$a^{(k+1)} = \frac{\mu}{K b^{(k)}}, \quad b^{(k+1)} = \frac{\nu}{K^{\top} a^{(k+1)}}, \quad (4)$$

and the coupling is given by $\pi = \text{diag}(a) K \text{diag}(b)$. Each iteration only requires matrix-vector products.

4 Results

On all our examples, the method preserves fine details while avoiding the aliasing artifacts observed with classical approaches.

Method	Points	Time (s)	Error
Direct correspondence	10^4	0.8	0.112
Sinkhorn (ours)	10^4	1.3	0.031
Sinkhorn (ours)	10^5	12.9	0.029

5 Conclusion

Optimal transport provides a natural interpolation between shapes of different topologies. We will next study the case of animated shapes.

References

- [1] C. Villani, *Optimal Transport: Old and New*, Springer, 2009.
- [2] M. Cuturi, Sinkhorn Distances, *NeurIPS*, 2013.